

THE REALIZATION OF VERDIER QUOTIENTS AS TRIANGULATED SUBFACTORS

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ABSTRACT. We give a method to realize the Verdier quotient as a subfactor of an arbitrary triangulated category. Our result includes a theorem of Buchweitz on the singularity category of a Gorenstein ring as a special case, and shows also that Iyama-Yoshino triangulated subfactors are Verdier quotients under suitable conditions.

1. INTRODUCTION

Let \mathcal{T} be a triangulated category and \mathcal{N} a triangulated subcategory of \mathcal{T} . Then the *Verdier quotient* \mathcal{T}/\mathcal{N} is obtained by formally inverting all morphisms s of \mathcal{T} such that the cone of s belongs to \mathcal{N} . The Verdier quotient is one of the most important tools for triangulated categories and has many important applications in algebra and geometry [3, 1, 8, 9].

In this paper, we use the homotopy theory of additive categories with suspensions developed by the author in [6, 7] to give a more simpler description of the Verdier quotient. The aim is to prove the following result which gives a unified, more general, proof of the recent work of Wei [10], Iyama-Yang [5] and Buchweitz [1] on (relative) singularity categories:

Theorem (3.2) *Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category and \mathcal{N} a triangulated subcategory. Assume that there is a \mathcal{N} -localization triple $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ in \mathcal{T} such that $(\mathcal{T}, [-1], [1], \mathbf{L}(\mathcal{C} \cap \mathcal{D}), \mathbf{R}(\mathcal{C} \cap \mathcal{D}), \mathcal{X})$ is a partial triangulated category. If $\mathcal{C} \cap \mathcal{D}$ is closed under extensions and \mathcal{X} is closed under direct summands, then there is a triangle equivalence*

$$(\mathcal{C} \cap \mathcal{D})/\mathcal{X} \xrightarrow{\sim} \mathcal{T}/\mathcal{N}$$

where $(\mathcal{C} \cap \mathcal{D})/\mathcal{X}$ is the triangulated subfactor category and \mathcal{T}/\mathcal{N} is the Verdier quotient of \mathcal{T} with respect to \mathcal{N} .

Our result shows also that Iyama-Yoshino triangulated subfactors are Verdier quotients under suitable conditions.

Throughout this paper, unless otherwise stated, that all subcategories of additive categories considered are full, closed under isomorphisms, all functors between additive categories are assumed to be additive.

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2. PRELIMINARIES

In this section we recall some basic facts and notions on the homotopy categories of additive categories with suspensions in [7].

Stable categories of additive categories. Let \mathcal{C} be an additive category and \mathcal{X} an additive subcategory of \mathcal{C} . We denote by \mathcal{C}/\mathcal{X} the *stable* or *factor category* of \mathcal{C} modulo \mathcal{X} . Recall that the objects of \mathcal{C}/\mathcal{X} are the objects of \mathcal{C} , and for two objects A and B the Hom space $\text{Hom}_{\mathcal{C}/\mathcal{X}}(A, B)$ is the quotient $\text{Hom}_{\mathcal{C}}(A, B)/\mathcal{X}(A, B)$, where $\mathcal{X}(A, B)$ is the subgroup of $\text{Hom}_{\mathcal{C}}(A, B)$ consisting of those morphisms factorizing through an object in \mathcal{X} . Note that the stable category \mathcal{C}/\mathcal{X} is an additive category and the canonical functor $\pi_{\mathcal{X}}: \mathcal{C}/\mathcal{X}$ is an additive functor. For a morphism $f: A \rightarrow B$ in \mathcal{C} , we use \underline{f} to denote its image in \mathcal{C}/\mathcal{X} under $\pi_{\mathcal{X}}$.

The localization of categories. We recall the definition of a localization of a category in [2]:

Definition 2.1. Let \mathcal{A} be a category and let \mathcal{S} be a class of morphisms of \mathcal{A} . The *localization of \mathcal{A} with respect to \mathcal{S}* means a category $\mathcal{A}[\mathcal{S}^{-1}]$ together with a functor $\gamma: \mathcal{A} \rightarrow \mathcal{A}[\mathcal{S}^{-1}]$ such that

- (i) $\gamma(s)$ is an isomorphism for each $s \in \mathcal{S}$, and
- (ii) whenever $F: \mathcal{A} \rightarrow \mathcal{B}$ is a functor carrying elements of \mathcal{S} to isomorphisms, there exists a unique functor $F': \mathcal{A}[\mathcal{S}^{-1}] \rightarrow \mathcal{B}$ such that $F' \circ \gamma = F$.

The second condition of Definition 2.1 shows that any two localizations of \mathcal{A} with respect to \mathcal{S} are canonically isomorphic. A general construction of the category $\mathcal{A}[\mathcal{S}^{-1}]$ is given by Gabriel and Zisman [2], but there is a foundational set-theoretic obstruction to its existence.

The homotopy category of additive categories with suspensions. Let \mathcal{A} be an additive category. Assume \mathcal{C}, \mathcal{D} and \mathcal{X} are three additive subcategories of \mathcal{A} such that $\mathcal{X} \subseteq \mathcal{C} \cap \mathcal{D}$. A morphism $f: A \rightarrow B$ in \mathcal{A} is said to be \mathcal{C} -*monic* if for each object $C \in \mathcal{C}$ the induced morphism $f^* = \text{Hom}_{\mathcal{A}}(f, C): \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$ is surjective. The notion of a \mathcal{C} -*epic* is defined dually. Recall that a morphism $f: A \rightarrow C$ in \mathcal{A} is called a \mathcal{C} -*preenvelope* (also called a *left \mathcal{C} -approximation* of A in the literature) if f is \mathcal{C} -monic and $C \in \mathcal{C}$. Dually a morphism $g: C \rightarrow A$ is called a \mathcal{C} -*precover* if g is \mathcal{C} -epic and $C \in \mathcal{C}$.

We denote by $\mathcal{C}^{\perp_{\mathcal{A}/\mathcal{X}}} = \{W \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}/\mathcal{X}}(\mathcal{C}, W) = 0\}$ and ${}^{\perp_{\mathcal{A}/\mathcal{X}}} \mathcal{D} = \{W \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}/\mathcal{X}}(W, \mathcal{D}) = 0\}$.

Definition 2.2. ([7, Definition 3.1]) The triple $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ is called a *localization triple* of \mathcal{A} if the following conditions hold:

- (a) For each $A \in \mathcal{A}$, there is a fixed sequence $W_A \xrightarrow{\omega_A} Q(A) \xrightarrow{r_A} A$ such that r_A is a \mathcal{C} -precover with ω_A as a weak kernel of r_A and $W_A \in \mathcal{C}^{\perp_{\mathcal{A}/\mathcal{X}}}$.
- (b) For each $A \in \mathcal{A}$, there is a fixed sequence $A \xrightarrow{j^A} R(A) \xrightarrow{\tau^A} W^A$ such that j^A is a \mathcal{D} -preenvelope with τ^A as a weak cokernel of j^A and $W^A \in {}^{\perp_{\mathcal{A}/\mathcal{X}}} \mathcal{D}$.
- (c) If $A \in \mathcal{D}$, then $Q(A) \in \mathcal{C} \cap \mathcal{D}$, and if $A \in \mathcal{C}$, then $R(A) \in \mathcal{C} \cap \mathcal{D}$.

If $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ is a localization triple, by [7, Remark 3.4], then the inclusion $\mathcal{C}/\mathcal{X} \hookrightarrow \mathcal{A}/\mathcal{X}$ has a right adjoint $Q: \mathcal{A}/\mathcal{X} \rightarrow \mathcal{C}/\mathcal{X}$ which sends an object A to $Q(A)$ and a morphism $\underline{f}: A \rightarrow B$ to $\underline{\check{f}}$ which satisfies $r_B \circ \check{f} = f \circ r_A$. Similarly, the inclusion $\mathcal{D}/\mathcal{X} \rightarrow \mathcal{A}/\mathcal{X}$ has a left adjoint $R: \mathcal{A}/\mathcal{X} \rightarrow \mathcal{D}/\mathcal{X}$ which sends an object A to $R(A)$ and a morphism $\underline{f}: A \rightarrow B$ to $\underline{\hat{f}}$ which satisfies $\hat{f} \circ j^A = j^B \circ f$.

Let $\text{Mor}(\mathcal{A})$ be the class of morphisms in \mathcal{A} . Then we define

$$(2.3) \quad \mathcal{S} = \{s \in \text{Mor}(\mathcal{A}) \mid RQ(\underline{s}) \text{ is an isomorphism in } (\mathcal{C} \cap \mathcal{D})/\mathcal{X}\}.$$

Then for each $A \in \mathcal{A}$, the morphism $r_A: Q(A) \rightarrow A$ is in \mathcal{S} and $j^A: A \rightarrow R(A)$ is in \mathcal{S} if $A \in \mathcal{C}$, and \mathcal{S} satisfies two out of three property (i.e. for two composable morphisms f, g in \mathcal{A} , two out of $f, f \circ g, g$ are in \mathcal{S} , so is the third). In this case, the category $\mathcal{A}[\mathcal{S}^{-1}]$ is called the *homotopy category* of the localization triple $(\mathcal{C}, \mathcal{X}, \mathcal{D})$.

We have the following theorem

Theorem 2.4. ([7, Theorem 3.8, Remark 3.9]) *Let $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ be a localization triple in an additive category \mathcal{A} . Then the homotopy category $\mathcal{A}[\mathcal{S}^{-1}]$ of $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ exists and is equivalent to the subfactor category $(\mathcal{C} \cap \mathcal{D})/\mathcal{X}$.*

Partial triangulated categories. Let \mathcal{A} be an additive category endowed with an additive endofunctor $\Sigma: \mathcal{A} \rightarrow \mathcal{A}$. We use $\mathcal{X} \subseteq \mathcal{C}$ to denote that \mathcal{X}, \mathcal{C} are additive subcategories of \mathcal{A} such that \mathcal{X} is a subcategory of \mathcal{C} .

A sequence of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ in \mathcal{A} is called a *right \mathcal{C} -sequence* if $C \in \mathcal{C}$, and g is a *weak cokernel* of f and h is a weak cokernel of g , i.e. the induced sequence

$$\text{Hom}_{\mathcal{A}}(\Sigma(A), \mathcal{A}) \rightarrow \text{Hom}_{\mathcal{A}}(C, \mathcal{A}) \rightarrow \text{Hom}_{\mathcal{A}}(B, \mathcal{A}) \rightarrow \text{Hom}_{\mathcal{A}}(A, \mathcal{A})$$

is exact. Dually, a sequence of the form $\Sigma(B) \xrightarrow{u} K \xrightarrow{v} A \xrightarrow{f} B$ in \mathcal{A} is called a *left \mathcal{C} -sequence* if $K \in \mathcal{C}$ and v is a weak kernel of f and u is a weak kernel of u .

Definition 2.5. ([6, Definition 2.2]) Let \mathcal{A} be an additive category endowed with an endofunctor Σ . Let $\mathcal{X} \subseteq \mathcal{C}$ be two additive subcategories of \mathcal{A} and $\text{R}(\mathcal{C})$ a class of right \mathcal{C} -sequences called *right \mathcal{C} -triangles*. The quadruple $(\mathcal{A}, \Sigma, \text{R}(\mathcal{C}), \mathcal{X})$ is said to be a *partial right triangulated category* if $\text{R}(\mathcal{C})$ is closed under isomorphisms and finite direct sums and the following axioms hold:

(PRT1) For each object $A \in \mathcal{C}$ there is a right \mathcal{C} -sequence $A \xrightarrow{i} X \rightarrow U \rightarrow \Sigma(A)$ in $\text{R}(\mathcal{C})$ with i an \mathcal{X} -preenvelope.

(ii) For each morphism $f: A \rightarrow B$ in \mathcal{C} , $A \xrightarrow{\begin{pmatrix} 1 \\ f \end{pmatrix}} A \oplus B \xrightarrow{(f, -1)} B \xrightarrow{0} \Sigma(A)$ is in $\text{R}(\mathcal{C})$.

(iii) If $A \xrightarrow{i} X \rightarrow U \rightarrow \Sigma(A)$ is in $\text{R}(\mathcal{C})$ with i an \mathcal{X} -preenvelope in \mathcal{C} , then for any morphism $f: A \rightarrow B$ in \mathcal{C} , there is a right \mathcal{C} -sequence $A \xrightarrow{\begin{pmatrix} i \\ f \end{pmatrix}} X \oplus B \rightarrow N \rightarrow \Sigma(A)$ in $\text{R}(\mathcal{C})$.

(PRT2) For any commutative diagram of right \mathcal{C} -sequences in $\text{R}(\mathcal{C})$

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & \Sigma(A) \\ \alpha \downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow \Sigma(\alpha) \\ A' & \longrightarrow & X & \xrightarrow{s} & U & \longrightarrow & \Sigma(A') \end{array}$$

with $X \in \mathcal{X}$, if α factors through f , then γ factors through s .

(PRT3) If the rows of the following diagram are in $\mathbf{R}(\mathcal{C})$ and the leftmost square is commutative, then there is a morphism $\gamma: C \rightarrow C'$ making the whole diagram commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma(A) \\ \alpha \downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow \Sigma(\alpha) \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma(A') \end{array}$$

(PRT4) If $A \xrightarrow{f} B \xrightarrow{l} C' \rightarrow \Sigma(A)$, $B \xrightarrow{g} C \xrightarrow{h} A' \xrightarrow{j} \Sigma(B)$ and $A \xrightarrow{g \circ f} C \xrightarrow{k} B' \rightarrow \Sigma(A)$ are in $\mathbf{R}(\mathcal{C})$ such that f and g are \mathcal{X} -monics in \mathcal{C} , then there is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{l} & C' & \longrightarrow & \Sigma(A) \\ \parallel & & \downarrow g & & \downarrow r & & \parallel \\ A & \xrightarrow{g \circ f} & C & \longrightarrow & B' & \longrightarrow & \Sigma(A) \\ & & \downarrow h & & \downarrow & & \downarrow \Sigma(f) \\ & & A' & \xlongequal{\quad} & A' & \xrightarrow{j} & \Sigma(B) \\ & & \downarrow j & & \downarrow & & \\ & & \Sigma(B) & \xrightarrow{\Sigma(l)} & \Sigma(C') & & \end{array}$$

such that the second column from the right is in $\mathbf{R}(\mathcal{C})$ with r an \mathcal{X} -monic.

The notion of a *partial left triangulated category* is defined dually.

Definition 2.6. Let $(\mathcal{A}, \Omega, \mathbf{L}(\mathcal{C}), \mathcal{X})$ be a partial left triangulated category and $(\mathcal{A}, \Sigma, \mathbf{R}(\mathcal{C}), \mathcal{X})$ a partial right triangulated category. The six-tuple $(\mathcal{A}, \Omega, \Sigma, \mathbf{L}(\mathcal{C}), \mathbf{R}(\mathcal{C}), \mathcal{X})$ is called a *partial triangulated category* if (Ω, Σ) is an adjoint pair and for each $A \in \mathcal{C}$, $\Omega(A) \xrightarrow{u} K \xrightarrow{v} X \xrightarrow{\pi} A$ is in $\mathbf{L}(\mathcal{C})$ with π an \mathcal{X} -precover if and only if $K \xrightarrow{v} X \xrightarrow{\pi} A \xrightarrow{-\psi_{A,K}(u)} \Sigma(K)$ is in $\mathbf{R}(\mathcal{C})$ with v an \mathcal{X} -preenvelope, where ψ is the adjunction isomorphism of (Ω, Σ) .

The main result for a partial triangulated category is the following:

Theorem 2.7. ([6, Theorem 6.3]) *Let $(\mathcal{A}, \Omega, \Sigma, \mathbf{L}(\mathcal{C}), \mathbf{R}(\mathcal{C}), \mathcal{X})$ be a partial triangulated category. Then the subfactor category \mathcal{C}/\mathcal{X} has a triangulated structure induced by $\mathbf{R}(\mathcal{C})$.*

For the convenient of the reader, we recall the induced triangulated structure by $\mathbf{R}(\mathcal{C})$ on \mathcal{C}/\mathcal{X} in [6, Section 3]. For each $A \in \mathcal{C}$, we fix a right \mathcal{C} -triangle $A \xrightarrow{i^A} X^A \xrightarrow{p^A} U^A \xrightarrow{q^A} \Sigma(A)$ with i^A an \mathcal{X} -preenvelope. Then define a functor $\Sigma^{\mathcal{X}}: \mathcal{C}/\mathcal{X} \rightarrow \mathcal{C}/\mathcal{X}$ by sending each object A to U^A and each morphism $\underline{f}: A \rightarrow B$ to $\underline{\kappa}^f$, where the morphism κ^f is defined by the following commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{i^A} & X^A & \xrightarrow{p^A} & U^A & \xrightarrow{q^A} & \Sigma(A) \\ f \downarrow & & \downarrow & & \downarrow \kappa^f & & \downarrow \Sigma(f) \\ B & \xrightarrow{i^B} & X^B & \xrightarrow{p^B} & U^B & \xrightarrow{q^B} & \Sigma(B) \end{array}$$

A *standard right triangle* in the subfactor category \mathcal{C}/\mathcal{X} is an induced sequence $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\xi(f,g)} \Sigma^{\mathcal{X}}(A)$ by the right \mathcal{C} -triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ with f an \mathcal{X} -monic in \mathcal{C} and the following

commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma(A) \\
\parallel & & \downarrow & & \downarrow \xi(f,g) & & \parallel \\
A & \xrightarrow{i^A} & X^A & \xrightarrow{p^A} & \Sigma^{\mathcal{X}}(A) & \xrightarrow{q^A} & \Sigma(A)
\end{array}$$

Denote by $\Delta^{\mathcal{X}}$ the class of the sequences in \mathcal{C}/\mathcal{X} which are isomorphic to the standard right triangles (the triangles in $\Delta^{\mathcal{X}}$ are called *distinguished triangles*). Then $(\Sigma^{\mathcal{X}}, \Delta^{\mathcal{X}})$ is the induced triangulated structure on \mathcal{C}/\mathcal{X} by $R(\mathcal{C})$ in Theorem 2.7.

3. THE PROOF OF THE MAIN RESULT

Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category. If \mathcal{C} is an additive subcategory of \mathcal{T} , then we will always denote by $R(\mathcal{C}) = \{A \xrightarrow{f} B \rightarrow C \rightarrow A[1] \in \Delta \mid C \in \mathcal{C}\}$ and $L(\mathcal{C}) = \{B[-1] \rightarrow K \rightarrow A \xrightarrow{f} B \in \nabla \mid K \in \mathcal{C}\}$. Recall that \mathcal{C} is said to be *extension-closed* if $A \rightarrow B \rightarrow C \rightarrow A[1]$ is in Δ such that $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.

Definition 3.1. Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category and \mathcal{N} a triangulated subcategory of \mathcal{T} . A localization triple $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ in \mathcal{T} is called a \mathcal{N} -localization triple if the following conditions hold:

- (a) For each $A \in \mathcal{T}$, the fixed sequences $W_A \xrightarrow{\omega_A} Q(A) \xrightarrow{r_A} A$ and $A \xrightarrow{j^A} R(A) \xrightarrow{\tau^A} W^A$ are parts of triangles in Δ .
- (b) $\mathcal{X} = \mathcal{C} \cap \mathcal{D} \cap \mathcal{N}$ and $\mathcal{C}^{\perp_{\mathcal{T}/\mathcal{X}}}, {}^{\perp_{\mathcal{T}/\mathcal{X}}} \mathcal{D} \subseteq \mathcal{N}$.

Theorem 3.2. Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category and \mathcal{N} a triangulated subcategory. Assume that there is a \mathcal{N} -localization triple $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ in \mathcal{T} such that $(\mathcal{T}, [-1], [1], L(\mathcal{C} \cap \mathcal{D}), R(\mathcal{C} \cap \mathcal{D}), \mathcal{X})$ is a partial triangulated category. If $\mathcal{C} \cap \mathcal{D}$ is closed under extensions and \mathcal{X} is closed under direct summands, then there is a triangle equivalence

$$(\mathcal{C} \cap \mathcal{D})/\mathcal{X} \xrightarrow{\sim} \mathcal{T}/\mathcal{N}$$

where $(\mathcal{C} \cap \mathcal{D})/\mathcal{X}$ is the triangulated subfactor category and \mathcal{T}/\mathcal{N} is the Verdier quotient of \mathcal{T} with respect to \mathcal{N} .

Proof. By assumption, the localization $\mathcal{T}[S^{-1}]$ of \mathcal{T} with respect to \mathcal{S} as defined in (2.3) exists and is equivalent to the subfactor category $(\mathcal{C} \cap \mathcal{D})/\mathcal{X}$ by Theorem 2.4. Since $(\mathcal{T}, [-1], [1], L(\mathcal{C} \cap \mathcal{D}), R(\mathcal{C} \cap \mathcal{D}), \mathcal{X})$ is a partial triangulated category, we know that $(\mathcal{C} \cap \mathcal{D})/\mathcal{X}$ has a triangulated structure induced by $R(\mathcal{C} \cap \mathcal{D})$ by Theorem 2.7.

Let \mathcal{S}' be a class $\{s: A \rightarrow B \mid \exists A \xrightarrow{s} B \rightarrow N \rightarrow A[1] \in \Delta \text{ with } N \in \mathcal{N}\}$ of morphisms of \mathcal{T} . Then the Verdier quotient \mathcal{T}/\mathcal{N} of \mathcal{T} with respect to \mathcal{N} is the localization of \mathcal{T} with respect to \mathcal{S}' . We will show that $\mathcal{S}' = \mathcal{S}$. In fact, since $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ is a \mathcal{N} -localization triple in \mathcal{T} , for each $A \in \mathcal{T}$, we have two triangles in Δ : $A[-1] \rightarrow W_A \xrightarrow{\omega_A} Q(A) \xrightarrow{r_A} A$ and $A \xrightarrow{j^A} R(A) \xrightarrow{\tau^A} W^A \rightarrow A[1]$ such that r_A is a \mathcal{C} -precover with $W_A \in \mathcal{N}$ and j^A a \mathcal{D} -preenvelope with $W^A \in \mathcal{N}$. Thus the morphisms r_A and j^A are in \mathcal{S}' . So for any morphism $s: A \rightarrow B$ in \mathcal{T} , $s \in \mathcal{S}'$ if and only if $\hat{s}: R(Q(A)) \rightarrow R(Q(B))$ is in \mathcal{S}' since \mathcal{S}' satisfies two out of three property. Similarly, $s \in \mathcal{S}$ if and only if \hat{s} is in \mathcal{S} . Thus in order to prove that $\mathcal{S} = \mathcal{S}'$, we may only consider the morphisms in $\mathcal{C} \cap \mathcal{D}$. Assume that we have a morphism $s: A \rightarrow B$ in $\mathcal{C} \cap \mathcal{D}$, extend it to a triangle $A \xrightarrow{s} B \rightarrow C \rightarrow A[1]$

in Δ . Since $(\mathcal{T}, [1], \mathbf{R}(\mathcal{C} \cap \mathcal{D}), \mathcal{X})$ is a partial right triangulated category, there is a right $\mathcal{C} \cap \mathcal{D}$ -triangle $A \xrightarrow{i^A} X^A \rightarrow U^A \rightarrow A[1]$ with $X^A \in \mathcal{X}$. By the cobase change of triangles in Δ , there is a commutative diagram of triangles:

$$\begin{array}{ccccccc}
 A & \xrightarrow{s} & B & \longrightarrow & C & \longrightarrow & A[1] \\
 i^A \downarrow & & \downarrow & & \parallel & & \downarrow \\
 X^A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & X^A[1] \\
 \downarrow & & \downarrow & & & & \\
 U^A & \xlongequal{\quad} & U^A & & & & \\
 \downarrow & & \downarrow & & & & \\
 A[1] & \longrightarrow & B[1] & & & &
 \end{array}$$

Since $\mathcal{C} \cap \mathcal{D}$ is closed under extensions we have $B' \in \mathcal{C} \cap \mathcal{D}$ and thus the triangle $A \xrightarrow{\begin{pmatrix} i^A \\ s \end{pmatrix}} X^A \oplus B \rightarrow B' \rightarrow A[1]$ is in $\mathbf{R}(\mathcal{C} \cap \mathcal{D})$ which induces a standard triangle $A \xrightarrow{\underline{s}} B \rightarrow B' \rightarrow \Sigma^{\mathcal{X}}(A)$ in $(\mathcal{C} \cap \mathcal{D})/\mathcal{X}$. Therefore \underline{s} is an isomorphism if and only if $B' \cong 0$ in $(\mathcal{C} \cap \mathcal{D})/\mathcal{X}$, and then if and only if $B' \in \mathcal{X}$. Since \mathcal{N} is a triangulated subcategory and $\mathcal{X} = \mathcal{C} \cap \mathcal{D} \cap \mathcal{N}$, $B' \in \mathcal{X}$ if and only if $C \in \mathcal{N}$. So s is in \mathcal{S}' if and only if it is in \mathcal{S} and we are done. Therefore the localization $\mathcal{T}[\mathcal{S}^{-1}]$ is just the Verdier quotient \mathcal{T}/\mathcal{N} . Next we will verify that the inclusion $E: (\mathcal{C} \cap \mathcal{D})/\mathcal{X} \hookrightarrow \mathcal{T}/\mathcal{N}$ is a triangle functor.

Applying the triangle functor $\gamma': \mathcal{T} \rightarrow \mathcal{T}/\mathcal{N}$ (the localization of \mathcal{T} with respect to \mathcal{S}') to $A \rightarrow X^A \rightarrow U^A \xrightarrow{q^A} A[1]$ we get an isomorphism $U^A \rightarrow A[1]$ in \mathcal{T}/\mathcal{N} , which defines a natural isomorphism $E \circ \Sigma^{\mathcal{X}} \simeq [1] \circ E$ by noting that $\Sigma^{\mathcal{X}}(A) = U^A$. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta} \Sigma^{\mathcal{X}}(A)$ be a distinguished triangle in $(\mathcal{C} \cap \mathcal{D})/\mathcal{X}$. Without loss of generality, we may assume that it is standard, i.e. there is a commutative diagram of triangles in Δ :

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\
 \parallel & & \downarrow & & \downarrow \eta & & \parallel \\
 A & \longrightarrow & X^A & \longrightarrow & U^A & \xrightarrow{q^A} & A[1]
 \end{array}$$

which shows that $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{q^A \circ \eta} A[1]$ is a distinguished triangle in \mathcal{T}/\mathcal{N} . Thus E is a triangle functor. \square

A triangle equivalence between Verdier quotients and Iyama-Yoshino triangulated subfactors. Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category. Recall that a pair $(\mathcal{U}, \mathcal{V})$ of additive subcategories of \mathcal{T} is called a *torsion pair* if $\text{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$ and for any object T in \mathcal{T} , there is a triangle $U \rightarrow T \rightarrow V \rightarrow U[1]$ in Δ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

A full triangulated subcategory of \mathcal{T} is called *thick* if it is closed under direct summands. Given a class \mathcal{C} of objects in \mathcal{T} , we use $\text{Thick}(\mathcal{C})$ to denote the smallest thick subcategory of \mathcal{T} containing \mathcal{C} , and denote by $\mathcal{C}^{\perp} = \{M \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(\mathcal{C}, M) = 0\}$ and ${}^{\perp}\mathcal{C} = \{N \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(N, \mathcal{C}) = 0\}$.

Lemma 3.3. *Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category and \mathcal{N} a triangulated subcategory of \mathcal{T} . If there are torsion pairs $(\mathcal{C}, \mathcal{C}^{\perp})$ and $({}^{\perp}\mathcal{D}, \mathcal{D})$ in \mathcal{T} such that $\mathcal{C}^{\perp}[-1] \subseteq \mathcal{D} \cap \mathcal{N}$, ${}^{\perp}\mathcal{D}[1] \subseteq \mathcal{C} \cap \mathcal{N}$, then $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ is a \mathcal{N} -localization triple in \mathcal{T} with $\mathcal{X} = \mathcal{C} \cap \mathcal{D} \cap \mathcal{N}$.*

Proof. Since $(\mathcal{C}, \mathcal{C}^\perp)$ and $({}^\perp\mathcal{D}, \mathcal{D})$ are torsion pairs in \mathcal{T} , by Corollary [6, Corollary 4.10], for all $A \in \mathcal{T}$, we can choose left \mathcal{T} -sequences $A[-1] \rightarrow W_A[-1] \rightarrow Q(A) \xrightarrow{\tau_A} A$ with $Q(A) \in \mathcal{C}$ and $W_A \in \mathcal{C}^\perp$, and right \mathcal{T} -sequences $A \xrightarrow{j^A} R(A) \rightarrow W^A[1] \rightarrow A[1]$ with $R(A) \in \mathcal{D}$ and $W^A \in {}^\perp\mathcal{D}$. Since $\mathcal{C}^\perp[-1] \subseteq \mathcal{D}$, ${}^\perp\mathcal{D}[1] \subseteq \mathcal{C}$, and \mathcal{C}, \mathcal{D} are closed under extensions, we know that Definition 2.2 (c) holds. Now assume there is a morphism $f: C \rightarrow G$ with $C \in \mathcal{C}, G \in \mathcal{C}^\perp[-1]$, then there is a left \mathcal{T} -sequence $G[-1] \rightarrow W_G[-1] \rightarrow Q(G) \xrightarrow{\tau_G} G$ with $Q(G) \in \mathcal{C}$ and $W_G \in \mathcal{C}^\perp$. Since \mathcal{C}^\perp is closed under extensions, we have $Q(G) \in \mathcal{C} \cap \mathcal{C}^\perp[-1] \subseteq \mathcal{X}$. Since r_G is a \mathcal{C} -epic, the morphism f factors through r_G , then $\text{Hom}_{\mathcal{T}/\mathcal{X}}(\mathcal{C}, \mathcal{C}^\perp[-1]) = 0$. Similarly we can prove that $\text{Hom}_{\mathcal{T}/\mathcal{X}}({}^\perp\mathcal{D}[1], \mathcal{D}) = 0$. Thus Definition 2.2 (a) and (b) hold. Therefore $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ is a localization triple in \mathcal{T} . By the above proof and the assumption $\mathcal{X} = \mathcal{C} \cap \mathcal{D} \cap \mathcal{N}$ we know that $(\mathcal{C}, \mathcal{X}, \mathcal{D})$ is a \mathcal{N} -localization triple. \square

Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category. Let \mathcal{X} and \mathcal{C} be additive subcategories of \mathcal{T} closed under direct summands. Assume that $(\mathcal{C}, \mathcal{C})$ forms an \mathcal{X} -mutation pair in the sense of [4, Definition 2.5]:

- (a) \mathcal{C} is extension-closed.
- (b) $\mathcal{X} \subseteq \mathcal{C}$ and $\text{Hom}_{\mathcal{T}}(\mathcal{X}[-1], \mathcal{C}) = 0 = \text{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{X}[1])$.
- (c) For any object $A \in \mathcal{C}$, there exists triangles $A[-1] \rightarrow K_A \rightarrow X_A \rightarrow A$ and $A \rightarrow X^A \rightarrow K^A \rightarrow A[1]$ in Δ such that $X_A, X^A \in \mathcal{X}$ and $K_A, K^A \in \mathcal{C}$.

Then by [6, Example 6.2 (ii)], $(\mathcal{T}, [-1], [1], \text{L}(\mathcal{C}), \text{R}(\mathcal{C}), \mathcal{X})$ is a partial triangulated category.

Corollary 3.4. *Let \mathcal{T} be a triangulated category. Let $\mathcal{X} \subseteq \mathcal{C}$ be two additive subcategories of \mathcal{T} such that $(\mathcal{C}, \mathcal{C})$ forms an \mathcal{X} -mutation pair. If $(\mathcal{C}, \mathcal{C}^\perp)$ is a torsion pair in \mathcal{T} and $\mathcal{X} = \mathcal{C} \cap \text{Thick}(\mathcal{C}^\perp)$, then there is a triangle equivalence*

$$\mathcal{C}/\mathcal{X} \xrightarrow{\sim} \mathcal{T}/\text{Thick}(\mathcal{C}^\perp)$$

where \mathcal{C}/\mathcal{X} is the Iyama-Yoshino triangulated subfactor and $\mathcal{T}/\text{Thick}(\mathcal{C}^\perp)$ is the Verdier quotient of \mathcal{T} with respect to $\text{Thick}(\mathcal{C}^\perp)$.

Proof. Take $\mathcal{N} = \text{Thick}(\mathcal{C}^\perp)$, $\mathcal{D} = \mathcal{T}$ in Corollary 3.3, we know that $(\mathcal{C}, \mathcal{X}, \mathcal{T})$ is a \mathcal{N} -localization triple, and therefore the assertion follows from Theorem 3.2. \square

Example 3.5. (Bousfield localization) Let \mathcal{T} be a triangulated category and \mathcal{C} a triangulated subcategory of \mathcal{T} . Assume that $(\mathcal{C}, \mathcal{C}^\perp)$ is a torsion pair in \mathcal{T} , take $\mathcal{X} = 0$ in Corollary 3.4, then we have a triangle equivalence $\mathcal{C} \simeq \mathcal{T}/\mathcal{C}^\perp$ which is just the Bousfield localization of \mathcal{T} with respect to \mathcal{C} .

A theorem of Wei. Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category. Given a subcategory \mathcal{C} of \mathcal{T} , we define the subcategory $\hat{\mathcal{C}}$ as the class of all objects T such that there are triangles $T_{i+1} \rightarrow C_i \rightarrow T_i \rightarrow T_{i+1}[1]$ in Δ for some $r \geq 0$ and all $0 \leq i \leq r$ satisfying $T_0 = T, T_{r+1} = 0$ and $C_i \in \mathcal{C}$ for each i . The subcategory $\check{\mathcal{C}}$ is defined dually. Let ω be a subcategory of \mathcal{T} . We use $\text{add}\omega$ to denote the class of all direct summands of finite direct sums of copies of objects in ω . Let \mathcal{X}_ω be the subcategory of \mathcal{T} consisting of all objects T satisfying that there are triangles $T_i \rightarrow M_i \rightarrow T_{i+1} \rightarrow T_i[1]$ in Δ for all $i \geq 0$, such that $T_0 = T$ and for all $i > 0$, $M_i \in \text{add}\omega$ and $\text{Hom}_{\mathcal{T}}(T_i, \omega[j]) = 0$ for all $j > 0$. Dually, one can define the subcategory ${}_\omega\mathcal{X}$. Following [10, Definition 2.4], an object in $\mathcal{G}_\omega = {}_\omega\mathcal{X} \cap \mathcal{X}_\omega$ is called ω -Gorenstein.

If ω is a semi-selforthogonal subcategory (i.e., $\text{Hom}_{\mathcal{T}}(\omega, \omega[i]) = 0$ for all $i > 0$) of \mathcal{T} and $\widehat{\mathcal{X}}_{\omega} = \mathcal{T} = \widetilde{\mathcal{X}}_{\omega}$, then $(\mathcal{X}_{\omega}, \text{add } \omega, {}_{\omega}\mathcal{X})$ is an $\langle \text{add } \omega \rangle$ -localization triple by [10, Lemma 2.2 and Lemma 2.6] (where $\langle \text{add } \omega \rangle$ is the triangulated subcategory of \mathcal{T} generated by $\text{add } \omega$). Moreover $(\mathcal{G}_{\omega}, \mathcal{G}_{\omega})$ is an $\text{add } \omega$ -mutation pair by [10, Proposition 2.5]. Thus $(\mathcal{T}, [-1], [1], \text{L}(\mathcal{G}_{\omega}), \text{R}(\mathcal{G}_{\omega}), \text{add } \omega)$ is a partial triangulated category by [6, Example 6.2 (ii)]. Therefore by Theorem 3.2 we have

Theorem 3.6. [10, Theorem 2.7] *There is a triangle equivalence $\underline{\mathcal{G}}_{\omega} \xrightarrow{\sim} \mathcal{T}/\langle \text{add } \omega \rangle$, where $\underline{\mathcal{G}}_{\omega}$ is the stable category of \mathcal{G}_{ω} modulo $\text{add } \omega$ and $\mathcal{T}/\langle \text{add } \omega \rangle$ is the Verdier quotient of \mathcal{T} with respect to $\langle \text{add } \omega \rangle$.*

Remark 3.7. The theorem above is a generalization of [5, Theorem 4.7] which is also in turn a generalization of [1, Theorem 4.4.1 (2)-(3)].

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